2 Dimensional Analysis and Dimensional Reasoning

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Abstract This chapter explores some of the ways physical dimensions, such as length, mass and time, impact on the work of scientists and engineers. Two main themes are considered: dimensional analysis, which involves deriving algebraic expressions to relate quantities based on their dimensions; and dimensional reasoning, a more general and often more subtle approach to problem solving. The method of dimensional analysis is discussed both in terms of its practical application (including the derivation of physical formulae, the planning of experiments, and the investigation of self-similar systems and scale models) and its conceptual contribution. The connection between dimensions and the fundamental concept of orthogonality is also described. In addition to these important uses of dimensions, it is argued that dimensional reasoning (using dimensionless comparisons to simplify models, the application of dimensional homogeneity to check for algebraic consistency, and the 'mapping-out' of solutions in terms of parameter space) forms the implicit foundation of nearly all theoretical work and plays a central role in the way scientists and engineers think about problems and communicate ideas.

2.1 Introduction

> …every particle of space is *always*, and every indivisible moment of duration is *everywhere*… (Isaac Newton 1687)

Measurement lies at the heart of both science and engineering. When we seek to make connections between our physical theories and experiment, or when we want to build a bridge or pumping station, we need to know the relationships between the quantities involved and - more specifically - their relative sizes. An experiment to test a theory of speed, for example, is not possible without some method of specifying relative distance, while a new railway bridge will be rightly considered a failure if it is not long enough to cover the relevant span. We are only able to practise science and engineering, therefore, by employing a system for communicating the relative magnitude of physical quantities. And it is this that necessitates measurement.

But what, exactly, does measurement involve? I could, if I so desired, use a ruler to measure the height in centimetres of the seat of my chair, finding this to be - say - 40cm. In the process I would be measuring a ratio: the ratio of the height of my chair to one centimetre, that is, 40:1. In doing so I have used a unit, in this case centimetres, as a base reference, but something more subtle has also taken place: I have made use of *dimensions*.

The dimensions of a measurement are independent of the units used as the basis of reference. Indeed, when measuring the height of my chair I could have used inches or feet, or even - should the distance have seemed especially large - light-years. Whatever the chosen unit, however, it would have to have been one appropriate for the dimension measured, in this case *length*. Exactly what length *is* constitutes something of an ontological problem, but for the time being I will state that it is the property of a system or object that confers upon that system or object extension in space. But length is not the only dimension of interest to a scientist or engineer. We must also make use of other dimensions; for example, *time*, which confers temporal extension, and *mass*. What should be noted here, as Newton observes in the opening quotation (Newton, 1687), is that each of these is in some sense fundamentally unique and cannot be expressed in terms of the others. And this difference, which allows the scientist or engineer to draw both distinctions between properties and to find relations between them, has profound consequences.

In this chapter I describe some of the important ways in which dimensions impact on, and are used in, the work of scientists and engineers. In doing so I will make a distinction between what I consider to be two key branches. First, there is *dimensional analysis*: explicit reasoning from the dimensional nature of elements in a set of physical quantities to algebraic relationships between those quantities; and second, what might be referred to as *dimensional reasoning*: the more implicit process of making comparisons between quantities of identical dimension, or of 'mapping out' a set of physical solutions in terms of dimensionless parameters. The first sense is possibly the one we are thinking of when we remember the *a priori* arguments taught at school or during undergraduate study. Nevertheless, the second, which makes up such an important part of the tacit skill base of those practising science and engineering, is perhaps that used most commonly.

What follows is broadly speaking divided along these lines. The latter half of our discussion will centre on a review of the method of dimensional analysis, consideration of its relationship to other forms of reasoning, and its application to both science and engineering. Though not intended as a primer in the method,[1] the inclusion here of a few worked examples will help to illustrate several discursive points. In principle, there is substantial philosophical work in this area. A thorough study of the issues involved - which range from 'fundamental dimensions', the place of dimensional argument in scientific explanation, to how and why such arguments work at all - is beyond the scope of this chapter, and will only be briefly touched on. However, for the purpose of trying to uncover just what a scientist

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[1] Plenty of excellent books exist on this topic, not least *Dimensional Analysis* (Huntley 1953) and *Dimensional Analysis for Engineers* (Taylor 1974)

or engineer does, it is perhaps dimensional reasoning that is most important, and the first half of our discussion shall begin here. Many of the themes in this section may at first appear to 'state the obvious', but they are also key elements of the cultural and working knowledge that underpins scientific research.

2.2 Dimensional reasoning

The dimensions length, mass and time, denoted by the letters L, M and T respectively, have already been introduced. In mechanical problems the set L, M and T form what we might call *fundamental dimensions*, so that the dimension of a given mechanical quantity must be expressed as some combination of these three. For example, when we consider the velocity of a particle v, we are interested in the number of length units the particle covers in a given number of time units. Thus the dimensions of velocity, written using the square bracket notation $[v] = V$, is length L divided by time T, that is, $V = LT^{-1}$. Because this expression gives the dimensions of V it is called the *dimensional formula* for V. Similarly, the dimensional formula for the particle's momentum $p = mv$, where m is its mass, is given by $[p] = MV = MLT^{-1}$, while the expression for its kinetic energy $E = \frac{1}{2}mv^2$ is $[E] = ML^2T^{-2}$.

Just which set of dimensions should be deemed generally fundamental is something of an open question. In the examples given above we could in principle take L, M and V as fundamental, yielding a dimensional formula for time $T = LV^{-1}$. That we don't probably has as much to do with the psychology of human beings, and our experiential understanding of the natural world from a 'common sense' perspective, as it does to the absolute nature of things. Indeed, it seems likely that scientists and engineers simply adopt whichever set seems most appropriate to the systems they study, just as they might choose an appropriate set of base units for measurement.[2]

Nevertheless, the choice of fundamental units has been a source of controversy, especially when that choice impacts on the solubility of a problem. In electrical systems, for instance, the necessity of introducing an additional dimension of charge Q is clear. However, in cases involving heat, defining a fundamental temperature dimension θ is more complicated. Indeed, there are some heat-flow problems that are only amenable to dimensional analysis provided θ is used in addition to L, M and T; yet from the perspective of kinetic theory, temperature is considered a form of kinetic energy with dimensions ML^2T^{-2}, rendering the same problem intractable (Rayleigh 1915a). As Lord Rayleigh notes, the situation is somewhat perplexing:

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[2] For a discussion of the history of dimensional analysis and the connection between fundamental dimensions and base units, see *The Mathematics of Measurement: A Critical History* by J. J. Roche (Roche 1998).

> It would indeed be a paradox if the further knowledge of the nature of heat afforded by molecular [kinetic] theory put us in a worse position than before in dealing with a particular problem. (Rayleigh 1915b)

Unfortunately Rayleigh doesn't offer a resolution, though he speculates that there is perhaps something qualitatively different about the nature of thermal kinetic energy which warrants the use of θ in certain contexts. From a kinetic perspective this has some justification: the thermal energy associated with a collection of particles is proportional to the *mean of the square* of their random velocities, not the square of their mean velocity. However, this argument on its own seems rather unsatisfactory. As Huntley suggests, it may be that ‘the criterion [for fundamentality] is purely pragmatic’ (Huntley 1953). Nevertheless, before considering the role of fundamentality in the method of dimensions, I shall review some of the important ways in which *dimensional reasoning* in general shapes the work of scientists and engineers.

2.2.1 Dimensional homogeneity and commensurability

Dimensional reasoning involves considering the relationship between physical quantities based on the dimensional properties of those quantities alone. It relies on the principle of *dimensional homogeneity*: every term in a physical equation must have the same dimensional formula. When a physical quantity q is expressed as a sum of n other quantities p_i, such that $q = p_1 + p_2 + \ldots + p_n$, this means $[q] = [p_i]$. Similarly, if a quantity q is proportional to the product of n other quantities p_i themselves raised to powers a_i, then

$$q = Cp_1^{a_1} p_2^{a_2} \cdots p_n^{a_n} \Rightarrow [q] = [p_1]^{a_1} [p_2]^{a_2} \cdots [p_n]^{a_n}, \qquad (0.1)$$

where C is a dimensionless constant of proportionality. Dimensional homogeneity makes intuitive sense to the extent that, for example, the total momentum of a system is the sum of constituent momenta in the system, not the constituent velocities or the constituent masses. In this way, dimensional reasoning comes into play whenever we use physical formulae and is perhaps its most prevalent application. Certainly, dimensional homogeneity is an invaluable tool in the day to day work of both scientists and engineers, and is employed whenever examining the dimensional consistency of terms in an equation (as a quick test for algebraic errors) or trying to recall half-forgotten formulae.

Reasoning of this kind is also essential when comparing quantities: only those with the same dimensional formula are commensurable. Making magnitude comparisons between quantities whose dimensions differ is not meaningful; for instance, there is no sense in which 2 kilograms are greater than 1 metre (length and mass are said to be *incommensurable*). However, knowing which variables dominate physical processes is important to our understanding of how a system operates. Fortunately, quantities can be compared if they are expressed as dimensionless ratios. In a perturbation analysis, for example, we might wish to consider the relative importance of two quantities δx and δy perturbed from corresponding variables x_0 and y_0. If the x and y variables have different dimensions this is not possible; however, finding dimensionless orderings, such as $\delta x/x \ll \delta y/y$, can reveal which effect is most important.

Similarly, we can make comparisons between units of identical dimension. And by forming derived units, such as characteristic time and length scales, scientists and engineers can employ dimensional commensurability to think about the validity of their models. For example, in a system of non-uniform temperature T_e we can use the dimensions of the gradient operator ∇, which are $[\nabla] = \mathrm{L}^{-1}$, to define a characteristic length-scale $l_T = T_e / |\nabla T_e|$ over which the temperature varies by T_e. This length scale may then be compared to other relevant length scales in the system. Indeed, a key parameter in plasma physics is the electron thermal mean-free-path λ_T, the mean distance an electron travels before undergoing a collision. Broadly speaking, if $\lambda_T < l_T$, then we may assume that electrons deposit the thermal energy associated with their motion locally: in a region at approximately the same temperature as the region from which they originated (Braginskii 1965). In this case a *local* model of heat transport may be used. On the other hand, if $\lambda_T > l_T$, then electrons will stream rapidly into regions where the temperature differs considerably from that of their origin and a *non-local* model of heat transport is necessary. The value of the dimensionless parameter l_T/λ_T may then be used by plasma physicists to quickly characterise the nature of the heat-flow in the systems they study.

2.2.2 Model simplification

A related use of dimensional comparisons concerns the simplification of mathematical expressions in theoretical work, which I shall illustrate using a contrived example, taken again from plasma physics. Consider the following expression for the rate of change of a vector function **f** in a fluid system with bulk flow velocity **C** and mean electron collision time τ_T :

$$\frac{\partial \mathbf{f}}{\partial t} = -\frac{\mathbf{f}}{\tau_T} + \left(\nabla \mathbf{C}\right) \cdot \mathbf{f}, \tag{0.2}$$

where $\nabla\mathbf{C}$ is a dyadic gradient operation yielding a rank-two tensor (matrix). Superficially, equations such as these may seem imposing due to the difficulty of interpreting the dyadic term; especially if the analytical form of **C** is either unknown or else fiendishly complicated. Nevertheless, we can make progress by combining physical understanding of the system and effective use of dimensional reasoning.

We proceed by noticing that for many systems the components of $\nabla\mathbf{C}$ are likely to be of a similar order of magnitude as $|\nabla C|$, with $C = |\mathbf{C}|$ as the plasma's bulk speed. This means that

$$\mathrm{O}\big((\nabla \mathbf{C})\cdot \mathbf{f}\big) \sim \mathrm{O}\big(C(|\nabla C|/C)\mathbf{f}\big) \tag{0.3}$$

where the symbols 'O' and '~' mean *'of the order'* and *'similar to'* respectively.

Furthermore, we observe using dimensional reasoning (as we did with temperature T_e length scales above) that the characteristic speed length scale is given by $l_C = C/|\nabla C|$. The quantity $t_C = l_C/C$ thus represents the time taken for the bulk plasma to traverse a distance equal to that over which the magnitude of C changes by C. Hence, with reference to equation (2.3), equation (2.2) may be written

$$\frac{\partial \mathbf{f}}{\partial t} = -\frac{\mathbf{f}}{\tau_T} + \mathrm{O}\left(\frac{\mathbf{f}}{t_C}\right) \Rightarrow \frac{\partial \mathbf{f}}{\partial t} \approx -\frac{\mathbf{f}}{\tau_T} \tag{0.4}$$

where the implied approximation is justified for many plasmas on the basis that the collision time τ_T is generally much shorter than t_C: that is, $\mathrm{O}(\mathbf{f}/t_C)$ may be neglected when compared with $\mathbf{f}/\tau_T$.

Crucially, by using our physical intuition, and a small degree of dimensional reasoning about the relevant length and timescales, we have successfully simplified an otherwise fairly opaque equation with relative ease. Whether or not equation (2.4) itself is soluble is not at stake here, the point is that we have made progress towards a solution. Such approaches are invaluable in theoretical work: exact expressions have a tendency to be mathematically intractable; approximate forms are often far more amenable to analysis.

2.3 Method of dimensional analysis

Having addressed some of the more general uses of dimensional reasoning, we are now in a position to explore the rather elegant topic of dimensional analysis proper. An example is instructive here and we shall introduce the method by considering the classic problem of a simple pendulum. This is a useful example for two reasons: firstly, it illustrates well the essential features of the method; and secondly, it highlights some of the method's conceptual peculiarities.

2.3.1 Analysis of the simple pendulum

We begin by formulating an abstracted model for the pendulum, and to do this physical reasoning and experience must first be employed to emphasise certain features of the system compared to others. This is a process situated at the core of mathematical modelling, and the technique used when deriving equation (2.4) above. The system is constructed as follows: a bob of mass m is suspended from a light string of length l and displaced so that the string and vertical define a small angle ϕ (see Fig. 2.1). Due to the restorative force of the bob's weight mg, where g

is the acceleration due to gravity, the pendulum will swing back and forth with a fixed time period t. It is an expression for t that we wish to obtain.

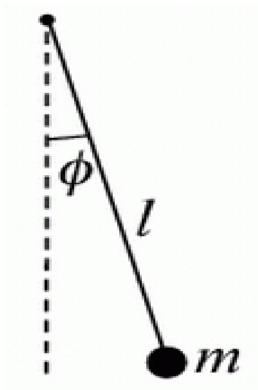

Fig. 2.1 The simple pendulum formed by suspending a bob of mass m on a string of length l. The angle ϕ between the string (solid line) and the vertical (dashed line) is assumed to be small.

The beauty of dimensional analysis is that this may be done without reference to either force balance or the solutions to differential equations. Indeed, the approach requires us simply to make an informed guess as to which quantities in the system are important and then proceed using basic arithmetic. Let us suppose, that t depends only on the pendulum's length, which has dimensions L, the bob's mass, which has dimensions M, and the acceleration due to gravity which has dimensions LT^{-2}.

Using what has been dubbed the *Rayleigh method*, we assume that t is proportional, by a dimensionless constant C, to the product of powers of l, m and g:

$$t = Cl^{\alpha}m^{\beta}g^{\gamma}, \tag{0.5}$$

where α, β and γ are constant exponents. Hence, by the expression for dimensional homogeneity of equation (2.1) we have:

$$[t]=[l]^{\alpha}[m]^{\beta}[g]^{\gamma} \Rightarrow \mathrm{T}=\mathrm{L}^{\alpha+\gamma}\mathrm{M}^{\beta}\mathrm{T}^{-2\gamma} \tag{0.6}$$

so that comparing the exponents on either side of equation (2.6) we find

exponents of T: $1 = -2\gamma$,
exponents of L: $0 = \alpha + \gamma$,
exponents of M: $0 = \beta$.

Thus, solving for α and γ to find $\alpha = -\gamma = 1/2$, and substituting these values into equation (2.5) we have

$$t = C\left(\frac{l}{g}\right)^{1/2} \tag{0.7}$$

Notice that dimensional analysis does not tell us the value of C (in this case 2π) which we would have to obtain from experiment or a different type of mathematical analysis; neither does it describe the importance of the angle ϕ, though alternative approaches do provide some insight (Huntley 1952). However, the method has revealed some important features of the system, such as the time period's proportionality to the root of the pendulum's length, inverse proportionality to the root of g, and independence on the mass of the bob m.

Indeed, dimensional argument differs from other forms of physical reasoning in a number of ways. It does not offer causal descriptions of phenomena and is unable, without supplementary experimentation, to furnish us with accurately predictive equations. Its status, therefore, as a means of explanation, or tool for dealing with physical problems requires some exploration.

2.3.2 Conceptual value

As a convenient route into a problem, especially when the underlying phenomenon is somehow obscured, dimensional analysis can prove unexpectedly powerful, and often enables scientists and engineers to bypass a more complicated mathematical treatment. Some of the classic applications of the method have been of this form: Rayleigh's initial exploration of why the sky is blue, for example, does not recourse to full-blown electrodynamic modelling of interactions between incident light from the sun and scattering particles in the upper atmosphere (Rayleigh 1871a; 1871b). Indeed, in the space of two short paragraphs he argues that the ratio of the amplitudes of scattered and incident light R_A is proportional to a dimensionless function of three quantities: its wavelength λ, which has dimensions L; the distance to the scattering particle r, which also has dimensions L; and the particle's volume V, which has dimensions L^3. The relationship $R_A \propto V/r$ was already known to Rayleigh, so that r, λ, and V may only be combined in a dimensionless fashion to give $R_A \propto (V/r\lambda^2)$ and the key result:

> ...the ratio of the amplitudes of the vibrations of the scattered and incident light varies inversely as the square of the wave-length, and the intensity of the lights themselves as the inverse fourth power. (Rayleigh 1871a)

Thus, since blue light has a wavelength approximately half that of red light, it is more strongly scattered than the latter by a factor of ~16: it is this blue scattered light that we see when looking at the sky.[3]

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[3] That we see blue light rather than purple is a result of additional effects, including the physiology of the human eye.

Indeed, though not able to provide complete solutions, the fact that dimensional analysis can supply key scalings is invaluable. In the example of the pendulum, the unknown constant of proportionality between t and $(l/g)^{1/2}$ is in many ways of less interest than the functional dependence of t on l and g. To the physicist or engineer a working knowledge of such dependence is often of more use than a numerically accurate expression. If, for example, we want to double the period of a given pendulum, equation (2.7) tells us that we must quadruple its length; this reveals more about pendulums generally than, after having l, g and the 2π proportionality supplied, calculating the value of t for a specific case.

Knowledge of relevant scaling is particularly important when trying to conserve experimental effort, as the following example adapted from Huntley demonstrates (Huntley 1952). We suppose that an experiment is designed to study the dependence of a quantity q on other variables r, s and t. During the experimental planning stage, a brief dimensional analysis yields

$$q = \frac{r}{t^2}\left(\frac{s}{t}\right)^{\alpha} \tag{0.8}$$

where α is an unknown exponent to be determined.

Our *a priori* expression for q tells us that an experiment to measure the change of q with r, though enabling us to determine the constant of proportionality, would not be the most fruitful choice of investigation. It would be much better, in this case, to consider the change of q with s or t, and in doing so determine the value of α. We might, for example discover that $\alpha = -2$, implying that q is independent of t and proportional to r/s^2. When detailed theoretical work is either intractable or obfuscatory, the method of dimensions can support the work of scientists and engineers, acting as a guide in experimental endeavour.

Nevertheless, dimensional analysis has an additional, and more subtle conceptual value prompting different ways of responding to a physical solution. In the pendulum example we began by explicitly selecting what we thought were the principle variables of the system, namely l, g and m, only to go on to exclude m from appearing in our expression for t. In this way, the analysis tells us very clearly that the time period of a pendulum *cannot* depend on m. Alternatively, had we begun by arguing from Newton's laws of motion, the cancellation of m in our derivation may well have passed unnoticed as an instance of mathematical felicity. In the first case we set out what we believed to be important, discovering both what *is* and what *is not* by comparison with our initial supposition. In the second case no such comparison is available because no initial supposition is made. Conceptually we learn different things about the system depending on our method of analysis, though these methods are, of course, complementary. By following the dimensional method we discover that there is no functional dependence of t on m; turning to a forces-based argument we can find out why.

2.3.3 Dimensions and orthogonality

Before proceeding further it is worth considering in greater detail how dimensional analysis and dimensional reasoning work, and what their working means for our understanding of physical problems. As we have seen, the basic principle of the analysis is dimensional homogeneity, but that this principle should be effective at all is a consequence of its relationship to one of the key concepts in science and engineering: orthogonality.

There are a number of ways of thinking about orthogonality. However, the one most appropriate to dimensional analysis and dimensional reasoning is the orthogonality of vectors (though it is not clear that dimensions themselves form a vector space in the proper sense). For example, displacement is a scalar quantity, with an absolute magnitude reflecting the distance displaced; while the change in position following a displacement is a vector quantity, reflecting the magnitude of displacement and direction in which the displacement occurs. In an orthogonal system the different components of a vector act independently. For example, following a displacement in a Cartesian (x, y) geometry, the change in the x-position tells us nothing about the change in y; information about how both vary is needed for a complete description.

In a similar fashion, the dimensions of a quantity vary independently. When we compared exponents of L, T and M in equation (2.6), we were able to do so because any exponent change in a given dimension on one side of the equation had to be accounted for by an equal change in the same dimension on the other. Indeed, this is the basis of dimensional homogeneity discussed earlier, which in some respects represents one of the most fundamental manifestations of orthogonality in natural science.

The connection between orthogonality and dimensions may be made more explicit by introducing the concept of *vector length*.[4] Treating length as a scalar we may denote its singular dimension as L; while treating it as a vector we find it has three component dimensions L_x, L_y and L_z, where the subscript carries information about the (Cartesian) direction considered. Furthermore, these definitions allow us to treat the dimensions of other quantities with greater descriptive precision. In a vector system, for example, velocity in the x-direction may be more accurately given the dimensions of L_xT^{-1}. And as we shall see in the following section, this approach greatly enhances the power of dimensional analysis.

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[4] Huntley provides a comprehensive development of this approach (Huntley 1952), attributing its origin to much earlier work in the late nineteenth century (Williams 1892).

2.4 Scale models, similarity and scaling laws

The method of dimensions can be particularly important to engineers during the process of designing prototypes, and has long been used as a technique for thinking about the relationship between scale models and what those models actually represent. Here it is the *principle of similitude* which is important; the idea that the responses of different sized models may be compared because relevant physical phenomena act in the same way over a range of scales. In the past, these approaches have been widely applied to problems involving the motion of bodies in fluids; systems for which the number of variables, and the complexity of the equations relating them, makes general solution impossible. Indeed, such cases are often of considerable technological importance: when designing an aircraft, for example, it is necessary to know the impact of air resistance when planning the number and power of its engines, and the size of its fuel tank.

Aircraft design is one of the classic applications of dimensional analysis, and serves as a useful example of how scaling laws derived from the method may be applied in a practical setting. To illustrate this we shall consider a slightly artificial scenario, again adapted from Huntley (Huntley 1952), making use of the vector length dimensions previously mentioned. Symmetry considerations, however, mean that Huntley's analysis can be simplified from the set of three vector lengths (L_x, L_y and L_z) to two.

Suppose that we wish to predict the resistance R likely to be met by a prototype aircraft using measurements derived from a scale model in a wind tunnel. Our first task is to find an expression for the air resistance to a model placed in a medium (air) of density ρ, viscosity η and flowing with velocity v. Since, the direction of flow defines only one unique axis (about which solutions are invariant under rotation) we require only two fundamental length dimensions: those parallel to flow $L_{\parallel}$ and those perpendicular $L_{\perp}$. In this way, v has dimensions $L_{\parallel}T^{-1}$, while the model's characteristic length l and cross-section A have dimensions $L_{\parallel}$ and $L_{\perp}^2$ respectively.[5] The quantities and their dimensions are tabulated below:

Quantity q	**Dimensions** $[q]$						
Resistance R	M	×	$L_{\parallel}$	-	-	×	T^{-2}
Flow velocity v	-	-	$L_{\parallel}$	-	-	×	T^{-1}
Density of medium ρ	M	×	$L_{\parallel}^{-1}$	×	$L_{\perp}^{-2}$	-	-
Characteristic length l	-	-	$L_{\parallel}$	-		-	-
Characteristic cross-section A	-	-	-	-	$L_{\perp}^{2}$	-	-
Viscosity of medium η	M	×	$L_{\parallel}^{-1}$	-	-	×	T^{-1}

[5] If flow is in the x-direction, for example, this may be thought of as setting $L_x = L_{\parallel}$ and $L_{\perp} = (L_yL_z)^{1/2}$, so that $[l] = L_{\parallel} = L_x$ and $[A] = L_{\perp}^2 = L_yL_z$.

Table 2.1 Physical quantities and their dimensions.

As before, we assume that R may be expressed as the product of the other quantities raised to fixed powers α, β, γ, χ and δ, multiplied by a dimensionless constant of proportionality C. In this way we have

$$R = Cv^{\alpha}\rho^{\beta}l^{\gamma}A^{\chi}\eta^{\delta} \tag{0.9}$$

Performing the analysis we find that R cannot be unequivocally determined, so that

$$R = Cv^2\rho A\left(\frac{\rho vA}{l\eta}\right)^{-\delta} \tag{0.10}$$

where δ is unknown.

The dimensionless quantity $\rho vA/l\eta$ is in fact Reynold's number, a key parameter in fluid mechanics. That δ should remain undetermined reflects the fact that scaling with $\rho vA/l\eta$ is dependent on the peculiarities of a given problem, a point we shall return to. However, for current purposes we note that comparison between the model and the prototype is possible without knowing δ provided we ensure $\rho vA/l\eta$ takes the same value in both systems. The process of enforcing identical values for dimensionless quantities in equations for systems on different scales is referred to as *dimensional similarity*, and may be understood as follows.

Denoting quantities associated with the model using the subscript m and those associated with the prototype using the subscript p, and assuming that the model is on a scale 1:r, the characteristic length scales and cross-sections are such that $l_p/l_m = r$ and $A_p/A_m = r^2$. Then, since the density and viscosity of air is the same for both model and prototype, that is, $\eta_m = \eta_p = \eta$ and $\rho_m = \rho_p = \rho$, we find

$$\frac{R_p}{R_m} = \left(\frac{v_p}{v_m}\right)^2 r^2 \left(\frac{\rho v_p A_p}{l_p\eta}\right)^{-\delta}\left(\frac{l_m\eta}{\rho v_m A_m}\right)^{-\delta}. \tag{0.11}$$

Notice that δ remains problematic because the dimensionless quantity $\rho vA/l\eta$ has not yet been fixed to take the same value on both scales. For this condition to hold, the model must be tested under conditions such that $v_m = rv_p$, in which case

$$\frac{\rho v_m A_m}{l_m\eta} = \frac{\rho\left(rv_p\right)\left(A_p/r^2\right)}{\left(l_p/r\right)\eta} = \frac{\rho v_p A_p}{l_p\eta} \tag{0.12}$$

and equation (2.11) reduces simply to the fraction

$$\frac{R_p}{R_m} = 1. \tag{0.13}$$

Hence, by measuring R_m we indirectly arrive at R_p. Indeed, what we have discovered is a *scaling law* between the model and the prototype: the resistance encountered by the prototype travelling with velocity v_p, is identical to that of the model in a wind tunnel with wind-speed $v_m = rv_p$. Explicitly stating the velocity argument in our notation for resistance, so that we use $R_p(v_p)$ for resistance in the former case and $R_m(v_m)$ for the latter, this scaling law may be expressed more concisely as

$$R_p(v_p) = R_m(v_m = rv_p). \tag{0.14}$$

Our example of the model aircraft is instructive because it clearly demonstrates the power of dimensional analysis when applied to problems of engineering in complicated systems such as fluids. It should be noted, however, that there is some artificiality to the problem in this instance. Given that aircraft must often travel at very high speeds, it may well be impossible to reach velocities of rv_p for the model. Even for large models, with $r \approx 10$ for example, wind-tunnel speeds of thousands of miles per hour would be necessary.

Nevertheless, aircraft designers can again rely on dimensional analysis for assistance. Supposing we found by experimentation with our model that $\rho vA/l\eta$ tends asymptotically towards a fixed value as v is increased. For sufficiently high speeds, therefore, we find that resistance becomes proportional to the square of the velocity, that is, equation (2.10) may be approximated as

$$R(v) \approx C'v^2\rho A, \tag{0.15}$$

where C' is the product of C with the asymptotic value of $\rho vA/l\eta$. Hence, combining this approximate expression with equation (2.14), it may be shown that

$$R_p(v_p) = \frac{R_m(rv_p)}{R_m(v_m)} R_m(v_m) \approx \left(\frac{v_p}{v_m}\right)^2 r^2 R_m(v_m). \tag{0.16}$$

In this way, investigation of the functional dependence of $\rho v_m A/l\eta$ with v_m provides us with a means of finding R_p even when wind tunnel speeds of rv_p are unobtainable.

However, the use of scaling laws and dimensional similarity is not restricted to design problems involving scale models. Indeed, in the 1940s Sir Geoffrey Taylor famously used dimensional scaling to discover the otherwise undisclosed yield of the United States' then most advanced atomic weapon (Taylor 1950a; 1950b).[6] Following its detonation at time t, in air of atmospheric density ρ, the energy E of a bomb may be related to the radius of its blast wave r using the Rayleigh method:

$$\begin{gathered}[E] = [r]^\alpha [t]^\beta [\rho]^\gamma \Rightarrow \mathrm{ML^2T^{-2}} = \mathrm{M^\gamma L^{\alpha-3\gamma} T^\beta} \Rightarrow E = C\frac{\rho r^5}{t^2} \\ \Rightarrow \log r = \frac{2}{5}\log t + \frac{1}{5}\log E - \frac{1}{5}\log\rho - \frac{1}{5}\log C\end{gathered} \tag{0.17}$$

Here I have disguised some of the complexity of Taylor's analysis (which involved solving a self-similar shock problem, not the simpler Rayleigh method) in

[6] These papers were not, so the story goes, popular with the U.S. military.

the unknown dimensionless quantity C.[7] The key point is that his solutions were dimensionally similar and he was able to determine $C \approx 1$ using scaling laws from smaller explosion trials of known energy. Labelled photographic snapshots of the atomic test were not secret (they had been declassified in 1947 and clearly indicated the radius of the blast in successive time frames), so it was then straightforward for Taylor to determine E from the intercept of the logarithmic plot of r against t. Taylor's estimate of the yield at 16,800 Tons is impressively close to the released figure of 20,000 Tons.

In more recent years, scaling laws have been successfully applied to experiment with laboratory astrophysics (see, for example, Falize et al. (2009)). Since actual astrophysical systems are well beyond the possibility of experimental replication on a 1:1 scale, dimensional similarity is a key method of investigation for scientists seeking repeatable tests of their theories.

2.4.1 The Π-Theorem

No discussion of the use of dimensional analysis would be complete without mentioning what has become known as the Π-Theorem - the seminal formal work on the topic published by E. Buckingham in the early 20th Century (Buckingham 1914). Broadly speaking, the theorem states that a dimensionally homogeneous equation involving n variables, defined in terms of k orthogonal dimensions, may be reduced to an equation in $(n - k)$ dimensionless parameters Π_1, Π_2, ..., Π_{n-k}. Together, these parameters characterise the system of interest and a relationship between them obtains such that:

$$\Pi_1 = \varphi_1\left(\Pi_2, \Pi_3, \cdots, \Pi_{n-k}\right), \tag{0.18}$$

where the function φ_1 is unknown and the subscripts are interchangeable. From this expression, further relations between the dimensionless quantities and their constituent variables may be derived in a similar fashion to the manner in which we proceeded earlier. For example, in the problem of the model aircraft we have six variables (R, v, ρ, l, η, A) and four orthogonal dimensions (M, $L_{\parallel}$, $L_{\perp}$, T), so that n = 6 and k = 4. The system may thus be characterised by the $n - k = 2$ parameters $\Pi_1 = (R/v^2\rho A)$ and $\Pi_2 = (\rho v A/l\eta)$. In this way equation (2.18) allows us to write

$$\frac{R}{v^2\rho A} = \varphi\left(\rho v A/l\eta\right), \tag{0.19}$$

which the reader may recognise as an alternative form of equation (2.10). Naturally there are subtleties to the Π-Theorem which have been glossed over here, but

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[7] I have also clearly used dimensional quantities inside the logarithms and the reader must assume that these are normalised by the units in which they are measured.

our result shows something important about the scope of dimensional analysis by making it clear that the functional dependence is not necessarily with exponent. Indeed, the undetermined value of δ in equation (2.10) was a consequence of this fact: δ is not simply a constant and the relationship of R to $(\rho vA/l\eta)$ is more complicated than a power law.

Nevertheless, as we saw with the scale model, the details of the function φ are not necessarily crucial so long as it may be approximated for a range of conditions. Indeed, the undetermined aspect of the analysis is often of benefit to scientists and engineers because it can reveal additional features of the underlying system. For the case considered, the dependence on $\rho vA/l\eta$ in a given regime will, more-or-less, follow a single power law; a shift towards a different power, therefore, reflects changes to the principal dynamics and transition to a different regime. This quality was born out in our treatment of model aircraft: the asymptotic approach of $\rho vA/l\eta$ towards a constant corresponds to the rapidly decreasing impact of viscosity, when compared to turbulence, in the high velocity regime (Huntley 1952).

2.4.2 Dimensionless parameters and solution space

The discussion of dimensionless parameters in the last topic returns us neatly to the more general consideration of dimensional reasoning in science and engineering that we began with. As we have seen, scientists and engineers typically use dimensionless parameters to 'map out' different regimes in the systems they study. These techniques become particularly powerful when equations are re-written in terms of dimensionless parameters and expressed graphically.

As an example, consider a plasma instability (Bissell 2010) whose maximum growth rate γ_M may be approximated by

$$\gamma_M = \varphi(\chi)\frac{1}{\tau_T}\left(\frac{\lambda_T}{l_T}\right)^2, \tag{0.20}$$

where λ_T and l_T are the electron thermal mean-free-path and temperature length scale in the plasma mentioned earlier, τ_T is the mean time between collisions and $\varphi(\chi)$ may be taken to be constant.[8] Suppose we wanted to graphically represent the growth-rate as a function of the temperature length scale. Naïvely, one might go about this by choosing values for λ_T and τ_T and plotting γ_M against l_T, thus showing the inverse proportionality inherent in equation (2.20). However, such a graph would only be immediately relevant to systems for which λ_T and τ_T took the same values as those used to construct the plot. It would be more interesting to use a dimensionless approach and plot the combined parameter $(\gamma_M\tau_T)$ against (l_T/λ_T);

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[8] These quantities were discussed in Sect. 2.2.1 *Dimensional homogeneity and commensurability* in the paragraph on derived units.

this continues to reveal the inverse proportionality of γ_M and l_T, but has the added advantage of being applicable to systems for which a vast range of values λ_T and τ_T are relevant. It should be noted that a local model was used to derive equation (2.20),[9] so that the l_T/λ_T axis has the additional advantage of naturally identifying areas where the expression for the growth rate begins to break down, namely, where $l_T/\lambda_T < 1$ for which $\lambda_T > l_T$.

The use of dimensionless parameters as the scales on axes in graphics is often referred to as mapping out the graphic in *parameter space*, and is a convenient means of displaying more information than may be relayed on other types of plot. It is a standard technique in the day to day work of scientists and engineers who seek to quickly spot regimes where given phenomena are important, or rapidly communicate their ideas to others during discussion.

2.5 Conclusions

Throughout this chapter we have seen a number of ways scientists and engineers use dimensional analysis in their work and research: be that through the *a priori* derivation of physical formulae; as an initial aid when planning fruitful experimentation; or the use of scale models, dimensional similarity and scaling laws. The power of the method rests not only in its relative ease of implementation, but also in its versatility; with applications ranging from problems involving heat-flow (Rayleigh 1915) to the study of a cat's lapping mechanism when drinking (Reis *et al*. 2010). Furthermore, the analysis can also impact on the way scientists and engineers interpret the systems they study, by identifying key combinations of variables as dimensionless parameters which characterise behaviour in different regimes.

However, alongside these varied and important applications it is worth remembering that dimensional analysis itself has its origins in the more humble, but no less significant, field of dimensional reasoning. Indeed, arguably the most frequent use of dimensions by scientists and engineers is reasoning of this form. When trying to remember half-forgotten formulae, or when attempting to derive new ones, the scientist or engineer will frequently appeal to the principle of dimensional homogeneity as an aid to memory or technique for identifying algebraic errors. In fact, every time an equation is written down, quantities compared and models simplified, the principle of dimensional homogeneity is tacitly employed. Furthermore, through the use of parameter space and dimensionless ratios, dimensional reasoning plays a key role in generalising concepts beyond specific circumstances enhancing the interpretation, discussion and communication of ideas.

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[9] See previous footnote.

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